1.0	
1.1	
1.25	1.4	1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NOSC

AD A112477

Technical Report 756

ANALYSIS OF AN ALGORITHM OF BITMEAD AND ANDERSON FOR THE INVERSION OF TOEPLITZ SYSTEMS

Harlan Sexton

7 January 1982

MAR 25 1982

DTIC FILE COPY

NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

82 03 25 018

NAVAL OCEAN SYSTEMS CENTER, SAN DIEGO, CA 92152

AN ACTIVITY OF THE NAVAL MATERIAL COMMAND

SL GUILLE, CAPT, USN
Commander

HL BLOOD
Technical Director

ADMINISTRATIVE INFORMATION

ACKNOWLEDGEMENTS

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NOSC Technical Report 756 (TR 756)	2. GOVT ACCESSION NO. AD-A 223	3. RECIPIENT'S CATALOG NUMBER 4·77
4. TITLE *(and Subtitle)* ANALYSIS OF AN ALGORITHM OF BITMEAD AND ANDERSON FOR THE INVERSION OF TOEPLITZ SYSTEMS		5. TYPE OF REPORT & PERIOD COVERED June and July 1981
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Harlan Sexton		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Ocean Systems Center San Diego CA 92152		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 7 January 1982
		13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Toeplitz forms
Matrix inversions
Displacement rank
Fast Fourier Transform (FFT)

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

In a paper entitled "Asymptotically Fast Solution of Toeplitz and Related Systems of Linear Equations," Bitmead and Anderson present an algorithm for the inversion of Toeplitz and related matrices. In this report their results are examined and some extensions are presented.
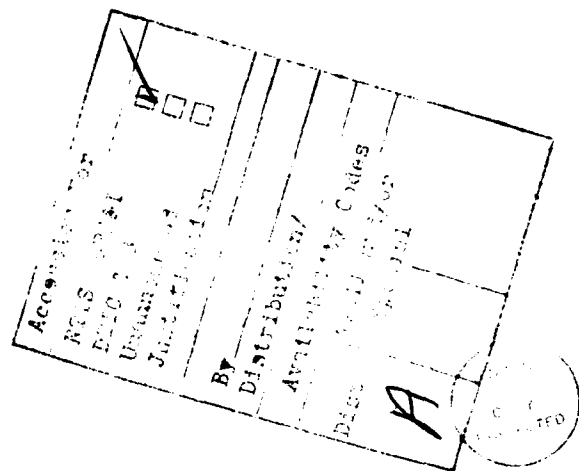
PREFACE

In a paper entitled "Asymptotically Fast Solution of Toeplitz and Related Systems of Linear Equations,"[1] Bitmead and Anderson present an algorithm for the inversion of Toeplitz and related matrices. In this report we shall examine their results and present some extensions.

This report, as well as Reference 1, has as a fundamental concept that of displacement rank of a matrix. (The proof of many of the results presented in this report and further results on displacement rank are presented in References 1 and 2.) The (+) displacement rank of a matrix may be defined as follows:

For M an n x n matrix given by

$$M = \begin{pmatrix} M_{1,1} \cdots M_{1,n} \\ \vdots \qquad \vdots \\ M_{n,1} \cdots M_{n,n} \end{pmatrix} \text{, define } M^+ \text{ by}$$

$$M^+ = \begin{pmatrix} 0 \quad \cdots \quad 0 \\ \vdots \; M_{1,1} \cdots M_{1,n-1} \\ \vdots \qquad \qquad \vdots \\ 0 \; M_{n-1,1} \cdots M_{n-1,n-1} \end{pmatrix}$$

The (+) displacement rank of M is equal to rank $(M-M^+)$. (The (-) displacement rank is obtained in an analogous way, except that M^- is obtained from M by taking "one step up the diagonal.") If M is a Toeplitz matrix, M is constant along diagonals and is, therefore, of displacement rank less than or equal to two. The displacement rank may be used as a measure of the "distance" of a matrix from being Toeplitz.

---

1. Bitmead, Robert R., and Anderson, Brian D.O., "Asymptotically Fast Solution of Toeplitz and Related Systems of Linear Equations," Linear Algebra and its Applications, 34, pp 103-116, 1980.

2. Kailath, Thomas, Kung, Sun-Yuan, and Morf, Martin, "Displacement Ranks of Matrices and Linear Equations," Journal of Mathematical Analysis and Applications, 68, pp 395-407, 1979.

A more direct, and more important, connection between displacement rank and Toeplitz matrices is the following: The (+) displacement rank of a matrix M is the smallest number α_+ so that M may be decomposed as $M = \sum_{i=1}^{\alpha_+} L_i U_i$, where L_i and U_i are lower and upper triangular Toeplitz matrices, respectively. (This will, in fact, be our definition and we shall show that the more natural definition given earlier is equivalent.) This type of decomposition is of value for two reasons. First, an upper or lower triangular Toeplitz matrix may be stored by storing a single vector. Second, the multiplication of two upper or two lower triangular matrices may be effected by a single convolution of two vectors. Thus, for large matrices of small displacement rank, such as large Toeplitz matrices, the matrices may be stored more economically in decomposed form. Further, as terms of the form UL can be decomposed economically in the form $L_1 U_1 + L_2 U_2 + L_3 U_3$, multiplication of large matrices with small displacement ranks may be performed economically. The details of obtaining and manipulating these decompositions form Section One of this report and the proofs of these results are presented in Section Two.

The reader may have noted that forming multiple products of the form $(L_1 U_1) \ldots (L_n U_n)$ will result in geometric growth in the number of terms in the final decomposition. This appears to be an intrinsic feature of such products. This unfortunate fact seems to be the major drawback to these techniques and, unfortunately, it is a very serious one. As we shall see in Section Four, the algorithm presented in Reference 1, with modifications for symmetric Toeplitz matrices has computational complexity which is $O(N\log_2^2 N)$ for N x N matrices. Due to the large number of terms in the decompositions of multiple products, the coefficient for this expression is approximately 6300. It is possible that this number might be smaller in practice, depending on statistical properties of the matrices to be inverted, but it seems likely that the algorithm will need substantial alteration before it becomes practicable.

This report is organized as follows: Section One contains a brief summary of the principle results needed to obtain and manipulate (+) and (-)

decompositions for matrices, together with a brief description of the algorithm presented in Reference 1. In Section Two the results of Section One, as well as some special results for symmetric matrices, are proved. Section Three is a detailed exposition of the algorithm given by Bitmead and Anderson, specialized for use in inverting symmetric, positive definite Toeplitz matrices. In Section Four the performance of this algorithm is analyzed.

While the algorithm of Reference 1 is of questionable interest, the subject of Toeplitz matrices is not, and the techniques presented in References 1 and 2 yield valuable insights into this subject. Beyond providing an analysis of an interesting "failure," it is hoped that this report can be of value as a collection (and clarification/correction) of results on the subject of displacement rank.

SECTION ONE

In this section the concept of displacement rank is defined and various results about it are stated. These results will be established in Section Two.

A. BASIC DEFINITIONS AND PROPERTIES

DEFINITION 1.1

The (+)-displacement rank of a matrix M is the smallest integer $\alpha_+(M)$ such that

$$M = \sum_{i=1}^{\alpha_+(M)} L_i U_i, \qquad (1.1)$$

where L_i and U_i are lower and upper triangular Toeplitz matrices, respectively. The (-)-displacement rank of M is defined as above with α_- replacing α_+ and L_i and U_i interchanged.

REMARK

It is obvious that for a Toeplitz matrix, T, $\alpha_+(T)$, $\alpha_-(T) \le 2$. This follows from the equation $T = T_\ell \cdot I + I \cdot T_u = I \cdot T_\ell + T_u \cdot I$, where T_ℓ and T_u are the upper and lower "parts" of T, respectively. It is not, perhaps, as obvious that every matrix has a finite displacement rank. This is not difficult to see from first principles, and it will follow immediately from Proposition 1.1 below.

DEFINITION 1.2

For a row vector $x=(x_1,\ldots,x_n)$, let $\tilde{x}=(x_n,\ldots,x_1)$. x^T will denote the column vector $\begin{pmatrix} x_1, \\ \vdots \\ x_n \end{pmatrix}$.

1

$L(x)$ denotes the lower triangular Toeplitz matrix with first column x^T. $U(x)$ denotes the upper triangular Toeplitz matrix with first row x. Let Z be the matrix $L((0,1,0,\ldots,0))$ and let Z' be the matrix $U((0,1,0,\ldots,0))$.

LEMMA 1.1

For an n x n matrix M, we have

    (a) $(ZMZ')_{ij} = M_{i-1,j-1}$ for i>1 and j>1.

                    $= \quad 0 \quad$ for i=1 or j=1.

    (b) $(Z'MZ)_{ij} = M_{i+1,j+1}$ for i<n and j<n.

                    $= \quad 0 \quad$ for i=n or j=n

REMARK

We see from Lemma 1.1 that ZMZ' is the displacement of M "one step down" the diagonal and that Z'MZ is the displacement "one step up."

PROPOSITION 1.1

Let M be an n x n matrix.

    (a) $\alpha_+(M) = \text{rank }(M-ZMZ')$.

    (b) $\alpha_-(M) = \text{rank }(M-Z'MZ)$.

    (c)   If M is invertible, then $\alpha_+(M) = \alpha_-(M^{-1})$.

    (d) $M-ZMZ' = \sum_{i=1}^{N} x_i^T y_i$ iff $M = \sum_{i=1}^{N} L(x_i)U(y_i)$.

    (e) $M-Z'MZ = \sum_{i=1}^{N} x_i^T y_i$ iff $M = \sum_{i=1}^{N} U(\bar{x}_i)L(\bar{y}_i)$.

2

REMARK

It is obvious from (a) and (b) of the above that the displacement ranks of M are both less than or equal to the dimension of M.

DEFINITION 1.3

Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$. x y is the row vector $z=(z_1,\ldots,z_n)$, where

$$z_j = \sum_{i=1}^{j} x_i y_{j+1-i}.$$

x#y is the row vector $w=(w_1,\ldots,w_{2n})$ given by $w=(x_1,\ldots,x_n,0,\ldots,0)*$ $(y_1,\ldots,y_n,0,\ldots,0)$. tr(x,i) is the n-vector $(x_1,\ldots,x_i,0,\ldots,0)$.

PROPOSITION 1.2

Let $x=(x_.,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$.

(a) $L(x)L(y)=L(x\ y)$.

(b) $U(x)U(y)=U(x\ y)$.

Let C_j^+ denote the $j\underline{th}$ column of $L(x)U(y)$.

Let C_j^- denote the $j\underline{th}$ column of $U(y)L(x)$.

Let R_i^+ denote the $i\underline{th}$ column of $L(x)U(y)$.

Let R_i^- denote the $i\underline{th}$ column of $U(y)L(x)$.

(c) $C_j^+(i) = \bar{x}\#tr(y,j)(n+j-i)$.

(d) $C_j^-(i) = y\#\overline{tr(x,n+1-j)}(n+j-i)$.

(e) $R_i^+(j) = \overline{tr(x,i)}\#y\ (n+j-i)$.

(f) $R_i^-(j) = tr(y,n+1-i)\#\bar{x}\ (n+j-i)$.

REMARK

Proposition 2 gives formulas for products of triangular Toeplitz matrices. Products of similar forms are given by a single convolution. For a product of dissimilar forms, an entire row or column of the product is obtained by one convolution.

PROPOSITION 1.3

Let x, y, $C_j^{\pm}$ and $R_i^{\pm}$ be as above.

(a) $L(x)U(y) = IL(z) + U(w)I - U(\tilde{x})L(\tilde{y})$,

where $\tilde{x} = (0, x_n, \ldots, x_2)$, $\tilde{y} = (0, y_n, \ldots, y_2)$, $z = R_n^{+}$, $w = \overline{tr(C_n^{+}, n-1)}$.

(b) $U(y)L(x) = L(z') \cdot I + I \cdot U(w') - L(\tilde{x})U(\tilde{y})$,

where $\tilde{x}$ and $\tilde{y}$ are as in (a), $z' = C_1^{-}$, and $w' = (0, R_1^{-}(2), \ldots, R_1^{-}(n))$.

REMARK

The contents of this result is that LU (or UL) products may be turned into UL (respectively, LU) products with two convolutions and some rearrangement of vector entries.

PROPOSITION 1.4

Let P be an $n \times n$ matrix of rank m, and let A be a nonsingular $m \times m$ minor. Let x_i^T be the column of P corresponding to the ith column of A, let w_k be the row of P corresponding to the kth row of A, and let W be the $m \times n$ matrix whose kth row is w_k. If y_j is the jth row of $A^{-1}W$, then

$$P = \sum_{i=1}^{m} x_i^T y_i.$$

4

(b) $\quad T_{11} = \sum_{i=1}^{\alpha_0} L((x_1^{(i)}, \ldots, x_n^{(i)}))\, U((y_1^{(i)}, \ldots, y_n^{(i)})).$

(c) $\quad T_{12} = \sum_{i=1}^{\alpha_0} L((x_1^{(i)}, \ldots, x_n^{(i)}))\, U((y_{n+1}^{(i)}, \ldots, y_{2n}^{(i)})) +$

$$L\left(\sum_{i=1}^{\alpha_0} (x_1^{(i)}, \ldots, x_n^{(i)}) \star (0, y_n^{(i)}, \ldots, y_2^{(i)})\right) \cdot I$$

(d) $\quad T_{21} = \sum_{i=1}^{\alpha_0} L((x_{n+1}^{(i)}, \ldots, x_{2n}^{(i)}))\, U((y_1^{(i)}, \ldots, y_n^{(i)})) +$

$$I \cdot U\left(\sum_{i=1}^{\alpha_0} (0, x_n^{(i)}, \ldots, x_2^{(i)}) \star (y_1^{(i)}, \ldots, y_n^{(i)})\right)$$

(e) $\quad T_{22} = \sum_{i=1}^{\alpha_0} L((x_{n+1}^{(i)}, \ldots, x_{2n}^{(i)}))\, U((y_{n+1}^{(i)}, \ldots, y_{2n}^{(i)})) +$

$L(v) \cdot I + I \cdot U(w).$

Here

$$v = \sum_{i=1}^{\alpha_0} (C_n^{+(i)}(n), \ldots, C_n^{+(i)}(2n-1))$$

and

$$w = \sum_{i=1}^{\alpha_0} (0, R_n^{+(i)}(n+1), \ldots, R_n^{+(i)}(2n-1)),$$

6

REMARK

Propositions 1.2 and 1.4 allow us, in principle at least, to form (+) or
(-) decompositions of arbitrary matrices M.

PROPOSITION 1.5

Let T be an invertible matrix subdivided into retangular blocks; T_{11},
T_{12}, T_{21}, and T_{22}, where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Let $S=T^{-1}$ be similarly subdivided. Then, if T_{11} is square and invertible,

(a) $S_{11} = T_{11}^{-1} + T_{11}^{-1} T_{12}(T_{22} - T_{21} T_{11}^{-1} T_{12})^{-1} T_{21} T_{11}^{-1}.$

(b) $S_{12} = -T_{11}^{-1} T_{12}(T_{22} - T_{21} T_{11}^{-1} T_{12})^{-1}.$

(c) $S_{21} = -(T_{22} - T_{21} T_{11}^{-1} T_{12})^{-1} T_{21} T_{11}^{-1}.$

(d) $S_{22} = (T_{22} - T_{21} T_{11}^{-1} T_{12})^{-1}.$

PROPOSITION 1.6

Let T be as in Proposition 1.5. Let each T_{ij} be square and of size
n x n. Let T have a (+) decomposition of the form

$$T = \sum_{i=1}^{\sigma_0} L((x_1^{(i)},\ldots,x_{2n}^{(i)})) \, U((y_1^{(i)},\ldots,y_{2n}^{(i)})).$$

Then

(a) S_{22} has (-) displacement rank less than or equal to α_0. Also,
the T_{ij} have (+)-decompositions of the following forms:

5

where $C_n^{+(i)}(j)$ is the <u>j</u>th entry of the <u>n</u>th column of $L(x^{(i)})$ $U(y^{(i)})$ and $R_n^{+(i)}(j)$ is the <u>j</u>th entry of the <u>n</u>th row of the same matrix.

B.   AN OUTLINE OF THE ALGORITHM

The algorithm described in Bitmead and Anderson[1], is a recursive one. Given a matrix of size 2n, the matrix is subdivided and the algorithm is applied to matrices of size n. These matrices are recombined to give the inverse of the 2n x 2n matrix. More precisely, the algorithm proceeds as follows:

Assume that $T = \displaystyle\sum_{i=1}^{\alpha_0} L(x_i)\, U(y_i)$ is given.

1.   Subdivide T into T_{11}, T_{12}, T_{21}, T_{22} using Proposition 1.6.

2.   Use the algorithm to write T_{11}^{-1} as $T_{11}^{-1} = \displaystyle\sum_{j=1}^{\alpha_0} U_j\, L_j$. (That such a

decomposition exists if T_{11} is invertible follows from Proposition 1.5(a) and Proposition 1.1(c).)

3.   Use the results of step 2, Propositions 1.2, 1.3 and 1.6 to write

$$(T_{22} - T_{21}\, T_{11}^{-1}\, T_{12}) = \sum_{k=1}^{\beta} L_k\, U_k.$$ (In fact, generally, the number

of terms is $3\alpha_0^3 + 6\alpha_0^2 + 4\alpha_0 + 2$.)

4.   Using Propositions 1.2, 1.4 and 1.1, express $(T_{22} - T_{21}\, T_{11}^{-1}\, T_{12})$ as

$\displaystyle\sum_{i=1}^{\alpha_0} \tilde{L}_i\, \tilde{U}_i$. (That this is possible follows from Proposition 1.6(a) and

Proposition 1.1(c).)

7

5. Use the algorithm and the result of step 4 to write S_{22} as

$$S_{22} = (T_{22} - T_{21} T_{11}^{-1} T_{12})^{-1} = \sum_{i=1}^{\alpha_o} U_i L_i.$$

6. Using the result of step 5 and step 2, calculate minors S_{11}, S_{12}, S_{21} of T^{-1}, where the notation is as in Proposition 1.5. Express each of the

minors as a sum of terms $S_{ab} = \sum_{i=1}^{\beta} U_i L_i$. (The number of terms depends on the subscripts "a" and "b." For S_{22}, $\beta = \alpha_o$. For S_{12} and S_{21}, $\beta = 3\alpha_o^3 + 3\alpha_o^2$. For S_{11}, $\beta = 9\alpha_o^5 + 18\alpha_o^4 + 9\alpha_o^3 + \alpha_o$.)

Using the results of step 6 and employing Propositions 1.1, 1.2 and 1.4, write S as

$$S = \sum_{i=1}^{\alpha_o} U_i L_i .$$

(This is possible by Proposition 1.1(c).)

In the next section, we shall prove the results quoted in this section, as well as stating and proving some special results for symmetric matrices. In the third section we shall give a more complete exposition of this algorithm for the special case of positive definite symmetric Toeplitz matrices.

In this section the proofs of Propositions 1.1-1.6 are given, as well as some special results for symmetric matrices.

LEMMA 1.1

For an n x n matrix M, we have

(a) $(ZMZ')_{ij} = M_{i-1,j-1}$ for i>1 and j>1.

$\qquad\qquad = 0$ for i=1 or j=1.

(b) $(Z'MZ)_{ij} = M_{i+1,j+1}$ for i<n and j<n.

$\qquad\qquad = 0$ for i=n or j=n.

PROOF

(a) Definition 1.2, $Z_{ik} = \delta_{i-1,k}, Z'_{mj} = \delta_{m,j-1}.$

Thus, $(ZMZ')_{ij} = \sum_{k,m=1}^{n} \delta_{i-1,k} M_{km} \delta_{m,j-1}$, which implies (a).

(b) $(Z'MZ)_{ij} = \sum_{k,m-1}^{n} \delta_{i,k-1} M_{km} \delta_{m-1,j}$, which implies (b).

LEMMA 2.1

Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$. If we define x_s, $y_t=0$ for s, t$\leq$0, then

(a) $[L(x) U(y)]_{ij} = \sum_{m=1}^{n} x_{i+1-m} y_{j+1-m}.$

(b) $[U(y) L(x)]_{k\ell} = \sum_{m'=1}^{n} y_{m'+1-k} x_{m'+1-\ell}.$

PROOF

From Definition 1.2,

$$[L(x)]_{i\ell} = x_{i+1-\ell} \quad \text{and} \quad [U(y)]_{kj} = y_{j+1-k},$$

with the convention that x_s, $y_t = 0$ for s, t≤0. From these two relations, (a) and (b) follow immediately.

LEMMA 2.2

Let M be an n x n matrix. If there exist row vectors $a_1, \dots, a_\ell$, $b_1, \dots, b_\ell$ such that

$$M = \sum_{i=1}^{\ell} a_i^T b_i,$$

then rank $M \leq \min(\text{rank}\{a_i\}, \text{rank}\{b_i\})$. Further, if rank(M) = m, then there exist row vectors $a_1, \dots, a_m$, $b_1, \dots, b_m$ such that the relation given above holds for M.

PROOF

As a linear operator on column vectors, the range of M is a subspace of span a_i^T and the range of M^T a subspace of span b_i^T. Since rank M = rank M^T, the inequality given above holds.

Next, if rank M = m, then there exists a spanning set of rows of M: $b_1, \dots, b_m$. If v_j is the jth row of M, then there exist coefficients $a_1^{(j)}, \dots, a_m^{(j)}$ such that

$$v_j = \sum_{i=1}^{m} a_i^{(j)} b_i \quad \text{for } 1 \leq j \leq n.$$

Let $a_i = (a_i^{(1)}, \dots, a_i^{(n)})$. Then the relation given above holds for M.

10

LEMMA 2.3

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and let $\bar{x} = (x_n, \ldots, x_1)$, $\bar{y} = (y_n, \ldots, y_1)$. Then

(a) $L(x)\,U(y) - Z\,L(x)\,U(y)\,Z' = x^T y$.

(b) $U(y)\,L(x) - Z'\,U(y)\,L(x)\,Z = \bar{y}^T \bar{x}$.

PROOF (a)

$$[L(x)\,U(y) - Z\,L(x)\,U(y)\,Z']_{ij} =$$

$$\sum_{m=1}^{n} x_{i+1-m}\,y_{j+1-m} - (1-\delta_{i,1})(1-\delta_{j,1}) \sum_{m=1}^{n} x_{i-m}\,y_{j-m},$$

by Lemmas 1.1 and 2.1. But this is just $x_i\,y_j$.

PROOF (b)

As above

$$(UL - Z'ULZ)_{k\ell} =$$

$$\sum_{m'=1}^{n} y_{m'+1-k}\,x_{m'+1-\ell} - (1-\delta_{k,n})(1-\delta_{\ell,n}) \sum_{m'=1}^{n} y_{m'-k}\,x_{m'-\ell}.$$

This equals $y_{n+1-k}\,x_{n+1-\ell}$, which is the $k\ell$ entry of $\bar{y}^T \bar{x}$.

LEMMA 2.4

If A and B are $n \times n$ matrices, $\mathrm{rank}(I-AB) = \mathrm{rank}(I-BA)$.

PROOF

Since A and B are $n \times n$, it is enough to show that the dimensions of the null spaces of $I-AB$ and $I-BA$ are equal. However, if $x_1, \ldots, x_k$ are independent

11

and satisfy $x_j = ABx_j$, then clearly $Bx_i,\ldots,Bx_j$ are independent (else $ABx_i = x_i$ are dependent) and $(Bx_j) = BA(Bx_j)$.

PROPOSITION 1.1

Let M be an n x n matrix.

(a) $\alpha_+(M) = \text{rank}(M - ZMZ')$.

(b) $\alpha_-(M) = \text{rank}(M - Z'MZ)$.

(c) If M is invertible, then $\alpha_+(M) = \alpha_-(M^{-1})$.

(d) $M - ZMZ' = \sum_{i=1}^{n} x_i^T y_i$ iff $M = \sum_{i=1}^{n} L(x_i) U(y_i)$.

(e) $M - Z'MZ = \sum_{i=1}^{n} x_i^T y_i$ iff $M = \sum_{i=1}^{n} U(\bar{x}_i) L(\bar{y}_i)$.

PROOF (d)

The map $A \to A - ZAZ'$ is linear. Also, by induction on the dimension, one can easily see that $A - ZAZ' = 0$ implies $A = 0$. It is clear from Lemma 2.3 that if $M = \sum L(x_i) U(y_i)$, that $M - ZMZ' = \sum x_i^T y_i$. Conversely, if $M - ZMZ'$ is equal to $\sum x_i^T y_i$, then let $M_0 = \sum L(x_i) U(y_i)$. Then $(M - M_0) - Z(M - M_0) Z' = 0$, and so $M = M_0$.

PROOF (e)

This follows just as for (d).

12

PROOF (a)

Say $M = \sum\limits^{\alpha_+} L(x_i)\, U(y_i)$. Then, by (d) and Lemma 2.2, $\mathrm{rank}(M - ZMZ') \leq \alpha_+$.

Also, if $\mathrm{rank}(M - ZMZ') = \beta$, then, by (d) and Lemma 2.2, $\beta \geq \alpha_+$.

PROOF (b)

This is immediate from (e) and Lemma 2.2.

PROOF (c)

If M is invertible, using Lemma 2.4, $\alpha_+(M) = \mathrm{rank}(M - ZMZ') =$
$\mathrm{rank}(I - (ZM)(Z'M^{-1})) = \mathrm{rank}(I - (Z'M^{-1})(ZM)) = \mathrm{rank}(M^{-1} - Z'M^{-1}Z) = \alpha_-(M^{-1})$.
(This is Theorem 1 of Kailath, Kung, and Morf, [2].)

PROPOSITION 1.2

Let $x = (x_1, \ldots, x_n)$, $y = y_1, \ldots, y_n)$.

(a) $L(x)\, L(y) = L(x\ y)$.

(b) $U(x)\, U(y) = U(x\ y)$. Let C_j^+ denote the j<u>th</u> column of $L(x)\, U(y)$.
    Let C_j^- denote the j<u>th</u> column of $U(y)\, L(x)$. Let R_i^+ denote the
i<u>th</u> row of $L(x)\, U(y)$. Let R_i^- denote the i<u>th</u> row of $U(y)\, L(x)$.

(c) $C_j^+(i) = \bar{x} \ \# \ \mathrm{tr}(y,j)\ (n+j-i)$.

(d) $C_j^-(i) = y \ \# \ \overline{\mathrm{tr}(x,n+1-j)}\ (n+j-i)$.

(e) $R_i^+(j) = \overline{\mathrm{tr}(x,i)} \ \# \ y(n+j-i)$.

(f) $R_i^-(j) = \mathrm{tr}(y,n+1-i) \ \# \ \bar{x}(n+j-i)$.

PROOF (a)

From Definition 1.2, $[L(x)]_{ij} = x_{i+1-j}$. Thus,

$$[L(x)\, L(y)]_{ij} = \sum_{k=1}^{n} x_{i+1-k}\, y_{k+1-j} = \sum_{k'=1}^{n} x_{k'}\, y_{(i+1-j)+1-k'} \, ,$$

by substituting $k'=i+1-k$. But this last is just the $(i+1-j)$ entry of $(x\, y)$, which is what we wished to show.

PROOF (b) •

As in (a)

$$[U(x)\, U(y)]_{ij} = \sum_{k=1}^{n} x_{k+1-i}\, y_{j+1-k} = \sum_{k'=1}^{n} x_{k'}\, y_{(j+1-i)+1-k'} \, ,$$

by substituting $k'=k+1-i$. This sum above is the $(j+1-i)$ entry of $(x\, y)$.

PROOF (c)

From Definition 1.3, for $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$,

$$tr(a,j) \,\#\, b(k) = \sum_{i=1}^{j} a_i\, b_{k+1-i}$$

letting $a=y$, $b=\tilde{x}$, $k=n+j-i$ and using the fact that $w\#v=v\#w$, we get that

$$\tilde{x} \,\#\, tr(y,j)(n+j-i) = \sum_{\ell=1}^{j} y_\ell\, x_{(n+1)-(n+j-i+1-\ell)}$$

$$= \sum_{\ell=1}^{j} y_\ell\, x_{i-j+\ell}$$

$$= \sum_{\ell'=1}^{n} y_{j+1-\ell'}\, x_{i+1-\ell'} = [L(x)\, U(y)]_{ij} \, .$$

The proofs of (d), (e) and (f) are similar to that given for (c).

14

PROPOSITION 1.3

Let x, y, $C_j^{\pm}$ and $R_i^{\pm}$ be as above.

(a) $L(x)\, U(y) = I \cdot L(z) + U(w) \cdot I - U(\tilde{x})\, L(\tilde{y})$,

where

$$\tilde{x} = (0, x_n, \ldots, x_2), \quad \tilde{y} = (0, y_n, \ldots, y_2),$$

$$z = \overline{R_n^+} \quad \text{and} \quad w = \overline{tr(C_n^+, n-1)}.$$

(b) $U(y)\, L(x) = L(z') \cdot I + I \cdot U(w') - L(\tilde{x})\, U(\tilde{y})$,

where

$\tilde{x}$ and $\tilde{y}$ are in (a), $z' = C_1^-$, and $w' = (0, R_1^-(2), \ldots, R_1^-(n))$.

PROOF (a)

For $i < n$ and $j < n$,

$$[L(x)\, U(y) - Z'L(x)\, U(y)Z]_{ij} =$$

$$C_j^+(i) - C_{j+1}^+(i+1) = \bar{x} \,\#\, tr(y,j)\, (u+j-i) -$$

$$\bar{x} \,\#\, tr(y,j+1)\, (n+j+1-i+1)$$

$$= -\bar{x} \,\#\, (0, \ldots, 0, y_{j+1}, 0, \ldots, 0)\, (n+j-i),$$

where the y_{j+1} occurs in the $j+1$ place. This last quantity is just $= -x_{i+1}$ y_{j+1}, by Definitions 1.2 and 1.3. (Recall that $\bar{x}_k = x_{n+1-k}$.) From this, the ij element of $LU - Z'LUZ$, for $i < n$ and $j < n$, is given by $-(x_2, x_3, \ldots, x_n, 0)^T$ $(y_2, \ldots, y_n, 0)$. The entries of the $\underline{nth}$ row are just given by $(0, 0, \ldots, 1)^T R_n^+$. The entries of the $\underline{nth}$ column (excluding the nn element, to avoid duplication) are given by $tr(C_n^+, n)^T (0, \ldots, 0, 1)$. Thus $LU - Z'LUZ$ is given by $(0, \ldots, 0, 1)^T R_n^+$

$+ \text{tr}(C_n^+,n)^T (0,\ldots,0,1) - (x_2,x_3,\ldots,x_n,0)^T (y_2,\ldots,y_n,0)$. The result then follows from Proposition 1.1(e).

PROOF (b)

This follows just as in (a).

PROPOSITION 1.4

Let P be an $n \times n$ matrix of rank m, and let A be a non-singular $m \times m$ minor. Let x_i^T be the column of P corresponding to the i<u>th</u> column of A, let w_k be the row of P corresponding to the k<u>th</u> row of A, and let W be the $m \times n$ matrix whose k<u>th</u> row is w_k. If y_j is the j<u>th</u> row of $A^{-1}W$, then

$$P = \sum_{i=1}^{m} x_i^T y_i \quad .$$

PROOF

Let $\underline{\bar{X}}$ be the $n \times m$ matrix whose i<u>th</u> column is x_i^T. Then, what we wish to show is, if we let $P' - \underline{\bar{X}} A^{-1} W$, that $P = P'$.

Let us examine the matrix $\underline{\bar{X}}$.

$$\underline{\bar{X}} = \begin{pmatrix} x_1^{(1)} & \ldots x_m^{(1)} \\ \vdots & \vdots \\ \boxed{x_1^{(r_1)} & \ldots x_m^{(r_1)}} \\ \vdots & \vdots \\ \boxed{x_1^{(r_m)} & \ldots x_m^{(r_m)}} \\ \vdots & \vdots \\ x_1^{(n)} & \ldots x_m^{(n)} \end{pmatrix}$$

16

Here, we have boxed the rows in $\underline{\bar{X}}$ which correspond to rows of A. (That is, the rows in P which contain the rows of A are labeled $r_1,\ldots,r_m$.) Similarly, the matrix W has the form

$$
W = \begin{pmatrix}
W_1^{(1)} \cdots & \boxed{W_1^{(C_1)}} & \cdots & \boxed{W_1^{(C_m)}} & \cdots & W_1^{(n)} \\
\vdots & \vdots & & \vdots & & \vdots \\
W_m^{(1)} \cdots & W_m^{(C_1)} & \cdots & W_m^{(C_m)} & \cdots & W_m^{(n)}
\end{pmatrix} ,
$$

where the m x m matrix formed by the boxed columns is also A.

It is immediate from its definition that the n x n matrix P' has rank$\leq$m. Further, it is clear from the above discussion of $\underline{\bar{X}}$ and W that the matrices P and P' agree on the ij elements where $i=r_1,\ldots,r_m$ <u>or</u> $j=C_1,\ldots,C_m$. We wish now to show that P=P'.

By interchanging rows and columns, we may take P and P' so that $r_i=i$ and $C_j=j$. Thus, we have

$$
P = \left(\begin{array}{c|c} A & W_1 \\ \hline \underline{\bar{X}}_1 & P_1 \end{array}\right) \quad \text{and} \quad P' = \left(\begin{array}{c|c} A & W_1 \\ \hline \underline{\bar{X}}_1 & P_1' \end{array}\right) ,
$$

where P and P' are both n x n and of rank m. Since the columns of A are independent, a column of W_1 may be expressed in only one way in terms of the columns of A. But this implies that the columns of P_1 and P_1' are equal, or P=P'.

LEMMA 2.5

Let P be an n x n matrix of rank m. If the ranks of the rows indexed by $i_1,\ldots,i_m$ and of the columns indexed by $j_1,\ldots,j_m$ are m, then the minor $(P_{i_k j_\ell})$ k,ℓ=1,...,m is invertible.

PROOF

This result is clearly independent of any row and column rearrangements, so that we may assume that $i_k=k$ and $j_\ell=\ell$. Let us denote the principle m x m minor of our (possibly rearranged) matrix P by M, and let us assume that M has rank k. Finally, let us take the columns of P to have been so arranged that the first k columns of M are a basis for the columns of M. Then P is of the form

$$P = \left(\begin{array}{c|c} M & R \\ \hline L & C \end{array} \right) \; ,$$

where M has rank k, (M|R) has rank m, $\left(\begin{array}{c} M \\ L \end{array}\right)$ has rank m and P has rank m. Let us suppose that k<m. Then, since the rank of (M|R) is m, there must exist a column of R which is independent of the columns of M. But then, the corresponding column of $\left(\begin{array}{c} R \\ C \end{array}\right)$ in P must be independent of $\left(\begin{array}{c} M \\ L \end{array}\right)$ in P, and thus the rank of P must be greater than m. This is a contradiction and so k=m and M is invertible.

PROPOSITION 2.1

Let P be a symmetric n x n matrix of rank m and let M be an invertible m x m minor. Then there exists an invertible m x m minor A, such that A is symmetric with respect to the diagonal of P and the rows of P which contain A also contain M.

PROOF

The rows of P which contain M have rank m. This is also true of those columns of P obtained by transposing these rows, since P is symmetric. The result follows from Lemma 2.5 if we take A to be the minor formed by the "intersections" of these rows and columns.

REMARK

This result will be of use to us in modifying the algorithm of Section 1 for the special case of T symmetric.

LEMMA 2.6

Let T be an invertible matrix subdivided into rectangular blocks T_{11}, T_{12}, T_{21}, T_{22}, where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} .$$

If T_{11} is invertible, then so is $T_{22} - T_{21}T_{11}^{-1}T_{12} = B$.

PROOF

Suppose that $T_{22}x = T_{21}T_{11}^{-1}T_{12}x$. Let $z = T_{11}^{-1}T_{12}x$. Then

$$T \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11}\ z \\ T_{21}\ z \end{pmatrix} = \begin{pmatrix} T_{12}\ x \\ T_{21}T_{11}^{-1}T_{12}\ x \end{pmatrix} = \begin{pmatrix} T_{12}\ x \\ T_{22}\ x \end{pmatrix} = T \begin{pmatrix} 0 \\ x \end{pmatrix} .$$

Since T is invertible, $x=0$.

REMARK

Lemma 2.6 is needed for the proof of Proposition 1.5.

PROPOSITION 1.5

For T, T_{ij} as in Lemma 2.7, let $S = T^{-1}$ and S be subdivided in a manner similar to that of T. Then

(a) $S_{11} = T_{11}^{-1} + T_{11}^{-1}T_{12}(T_{22} - T_{21}T_{11}^{-1}T_{12})^{-1}\ T_{21}T_{11}^{-1}$

(b) $S_{12} = -\ T_{11}^{-1}T_{12}(T_{22} - T_{21}T_{11}^{-1}T_{12})^{-1}$

(c) $S_{21} = -\ (T_{22} - T_{21}T_{11}^{-1}T_{12})^{-1}T_{21}T_{11}^{-1}$

(d) $S_{22} = (T_{22} - T_{21}T_{11}^{-1}T_{12})^{-1}$

PROOF

This follows easily from Lemma 2.7 by matrix multiplication.

PROPOSITION 1.6

Let T be as in Proposition 1.5 with each of the T_{ij} of size $n \times n$. Let T have a (+)-decomposition of the form

$$T = \sum_{i=1}^{\alpha_0} L((x_1^{(i)},\ldots,x_{2n}^{(i)}))\, U((y_1^{(i)},\ldots,y_{2n}^{(i)})).$$

Then

(a) S_{22} has (-)-displacement rank $\leq \alpha_0$. Also, the T_{ij} have
(+)-decompositions of the following forms:

(b) $$T_{11} = \sum_{i=1}^{\alpha_0} L((x_1^{(i)},\ldots,x_n^{(i)}))\, U((y_1^{(i)},\ldots,y_n^{(i)})).$$

(c) $$T_{12} = \sum_{i=1}^{\alpha_0} L((x_1^{(i)},\ldots,x_n^{(i)}))\, U((y_{n+1}^{(i)},\ldots,y_{2n}^{(i)}))\ +$$

$$L\left(\sum_{i=1}^{\alpha_0} (x_1^{(i)},\ldots,x_n^{(i)})\, (0,y_n^{(i)},\ldots,y_2^{(i)})\right)\ \cdot I.$$

(d) $$T_{21} = \sum_{i=1}^{\alpha_0} L((x_{n+1}^{(i)},\ldots,x_{2n}^{(i)}))\, U((y_1^{(i)},\ldots,y_n^{(i)}))\ +$$

$$I\cdot U\left(\sum_{i=1}^{\alpha_0} (0,x_n^{(i)},\ldots,x_2^{(i)})\, (y_1^{(i)},\ldots,y_n^{(i)})\right)\ .$$

(e) $$T_{22} = \sum_{i=1}^{\alpha_0} L((x_{n+1}^{(i)},\ldots,x_{2n}^{(i)}))\, U((y_{n+1}^{(i)},\ldots,y_{2n}^{(i)}))\ +$$

$$L(V)\cdot\ +\ I\cdot U(W).$$

20

Here

$$v = \sum_{i=1}^{\alpha_0} (C_n^{+(i)}(n),\ldots,C_n^{+(i)}(2n-1)) \qquad \text{and}$$

$$w = \sum_{i=1}^{\alpha_0} (0,R_n^{+(i)}(n+1),\ldots,R_n^{+(i)}(2n-1)),$$

where $C_n^{+(i)}(j)$ is the jth entry of the nth column of $L(x^{(i)}) U(y^{(i)})$ and $R_n^{+(i)}(j)$ is the jth entry of the nth row of the same matrix.

PROOF (a)

S_{22} is the lower right hand minor of size n x n in the 2n x 2n matrix S. Since S has (-)-displacement rank α_0, by Proposition 1.1(c), rank $(S-Z'SZ)=\alpha_0$. But $S_{22}-Z'S_{22}Z$ is a minor of that matrix and, therefore, $\alpha_+(S_{22}) \le \alpha_0$. (Here Z,Z' are assumed to be of the same size as the matrices S,S_{22} in expressions where they occur.)

PROOF (b)

To prove (b)-(e), it is clearly enough to show these results for $\alpha_0=1$. Consider the matrix $T=L(x_1,\ldots,x_{2n}) U(y_1,\ldots,y_{2n})$. The minor T_{11} is clearly the product of the first n rows of L with the first n columns of U, which is simply the expression in (b).

PROOF (c)

T_{12} is the product of the first n rows of L with the last n columns of U. However, since the first n rows of L are zero from the (n+1) entry on, $T_{12} = L_{11} \cdot U_{12}$, where the notation has the obvious meaning. Since U_{12} is of the form $U((y_{n+1},\ldots,y_{2n}) + L((0,y_n,\ldots,y_2))$, and since the product of lower triangular Toeplitz matrices is lower triangular Toeplitz, we have (c).

21

PROOF (d)

Follows just as does (c).

PROOF (e)

Consider $(LU-ZLUZ')_{22}$. This is just $(T_{22}-ZT_{22}Z')-R$, where R is the matrix whose first row is vector $(R_n^+(n),\ldots,R_n^+(2n-1))$, whose first column is $(C_n^+(n),\ldots,C_n^+(2n-1))^T$, and the rest of whose entries are zero. The formula in (e) follows from this and Proposition 1.1(d). (See Figure 1.)

$$
\text{If } A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) = \left(\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n,n} & \cdots & a_{n,n} & a_{n,n+1} & \cdots & a_{n,n+m} \\ \hline a_{n+1,n} & \cdots & & a_{n+1,n+1} & \cdots & a_{n+1,n-m} \\ \vdots & & & \vdots & & \vdots \\ a_{n,1} & \cdots & & a_{n+1,n+m} & \cdots & a_{n+m,n+m} \end{array}\right) \text{, then}
$$

$$
(A - ZAZ') = \left(\begin{array}{c|c} A_{11} - ZA_{11}Z' & A_{12} - ZA_{12}Z' \\ \hline A_{21} - ZA_{21}Z' & A_{22} - ZA_{12}Z' \end{array}\right) - \left(\begin{array}{c|c} & \begin{array}{c} 0 \\ a_{1n} \\ \vdots \\ a_{n-1n} \end{array} \\ 0 & \quad 0 \\ \hline 0\,a_{n,n}\cdots a_{n,n-1} & a_{n,n}\cdots a_{n,n+m-1} \\ 0 & \begin{array}{c} \vdots \\ a_{n,n+m-1} \end{array} \quad 0 \end{array}\right)
$$

Figure 1.

22

SECTION THREE

In this section a version of the algorithm outlined in Section One will be presented in more detail. We shall be concerned with the special case of T a symmetric positive definite Toeplitz matrix. In describing the algorithm, we shall make use of three "procedures," which we shall refer to as SUBDIVIDE, DECOMP and PRODUCT. We give descriptions of these procedures below, together with discussions of their storage requirements and computational complexity. Following the descriptions of the procedures, we present the algorithm.

A.   THE PROCEDURES

(1)   SUBDIVIDE

This procedure takes a (+)-decomposition (of a 2n x 2n matrix T) of the form $L(x_1) U(y_1) + L(x_2) U(y_2)$ and returns (+)-decomposition of the 4 n x n minors described in Proposition 1.6. The lengths of the decompositions of T_{11}, T_{12}, T_{21}, and T_{22} are, respectively, 2, 3, 3 and 4. If $\theta(n)$ is the number of computations needed to convolve two n-vectors (here we make no distinction between x*y and x#y), this procedure requires $4\theta(n)$ computations, plus some overhead which we shall ignore.

(Here we are not distinguishing between the various types of arithmetic operations in our analysis.)

The space required to store the "matrix" T goes up from 8n to 24n as a result of this procedure, assuming that the original decomposition of T is <u>not</u> saved.

(2)   DECOMP

Let R denote a symmetric positive definite n x n matrix with (+)(or(-))- displacement rank ≤ 2. This procedure takes a (+)(or(-))-decomposition of arbitrary length N, and returns a minimal (+)(or(-))-decomposition. Since the (+) and (-) versions of the procedure are completely analogous, we shall examine only the (+) version.

Let $P=R-ZRZ'$. By our hypothesis on R, P is symmetric, has rank ≤ 2, and the element P_{11} is positive, which implies that the first row and first column of P are both non-zero. Thus, if we wish to find a maximal minor of P, by Proposition 2.1 this minor is either of the form (P_{11}) (and the rank of P is 1), or of the form

$$\begin{pmatrix} P_{11} & P_{1\ell} \\ \hline P_{\ell 1} & P_{\ell\ell} \end{pmatrix} .$$

Therefore, all that is necessary is to calculate the determinants of 2 x 2 minors of the form given above. Thus, we need to calculate the first row and the diagonal elements of P.

If

$$R = \sum_{i=1}^{N} L(x_i)\, U(y_i), \quad \text{then } P = \sum_{i=1}^{N} x_i^T y_i ,$$

and calculating the first row of P requires N·n multiplications and N·n additions. Similarly, calculating the diagonal of P takes 2N·n computations. Calculation of the appropriate determinants requires 3n computations and comparison of the results to select the "best" minor requires approximately and additional n calculations. Finally, calculation of the minimal decomposition takes additional 6n computations if P has rank 2 and n computations if P has rank 1. Thus, the total number of computations is at most (2N + 10)·n. The required storage goes from 2Nn to at most 4n. (In the event that this algorithm or one like it is ever implemented, this procedure will need to be examined for numerical stability. It seems likely that this procedure is the critical area of the algorithm in this regard.)

(3) PRODUCT

Let D_i^+, D_j^- denote decompositions of (+), respectively (−), type. This procedure will have 3 versions: (1) 3-plus, (2) 3-minus, and (3) 5-minus. The 3-plus version takes decompositions D_1^+ and D_1^-, of ranks 3 and 2, respectively, and returns a (+) decomposition of the product $(D_1^+)(D_1^-)(D_1^+)$.

The 3-minus version takes decompositions D_1^+ and D_1^-, of ranks 3 and 2, respectively, and returns a (-)-decomposition of the product $(D_1^-)(D_1^+)(D_1^-)$. Finally, the 5-minus version takes decompositions D_1^+, D_1^-, and D_2^-, of ranks 3, 2, and 2, respectively, and returns a (-)-decomposition of the product $(D_1^-)(D_1^+)(D_1^-)(D_1^+)(D_1^-)$. As the three versions are analogous, we shall examine only the 3-plus version and state the results needed for the other two.

In evaluating the product $(D_1^+)(D_1^-)(D_1^+)$, there will be 3 x 2 x 3 = 18 terms of the form $(L_1U_1)(U_2L_2)(L_3U_3)$ to decompose. Using Proposition 1.2 and 2 x 18 = 36 convolutions, these terms become 18 terms of the form $L_1(U_4L_4)U_3$. Then, by 2 x 18 = 36 more convolutions we obtain, from Proposition 1.3, 18 terms of the form $L_1(L_5 \cdot I + I \cdot U_5 + L_6U_6)U_3$. This requires an additional 4 x 18 convolutions to yield, finally, 3 x 18 = 54 terms of the form L_7U_7. Thus, the total number of convolutions required is 36 + 36 + 72 = 144, and the storage requirement is an additional 108n.

The 3-minus version requires 96 convolutions and space for an additional 72n numbers.

The 5-minus version requires 2016 convolutions and space for an additional 648 terms of the form UL, or 1296n additional numbers.

B. THE ALGORITHM

The algorithm for a symmetric, positive definite, 2^k x 2^k Toeplitz matrix T is as follows:

        (0) Assume that T has been decomposed as $L_1U_1 + L_2U_2$.

        (1) If the input matrix is of sufficiently small size, invert it. Otherwise, call SUBDIVIDE.

        (2) Use the algorithm to invert T_{11} and return T_{11}^{-1} as UL + U'L'.

        (3) Use the results of SUBDIVIDE, step (2), and the 3-plus version of PRODUCT to form a (+)-decomposition of $T_{21}T_{11}^{-1}T_{12}$. Adjoin the decomposition of T_{22} given by SUBDIVIDE to form a (+)-decomposition of the matrix $B = T_{22} - T_{21}T_{11}^{-1}T_{12}$. (Since B is a symmetrically located minor of the symmetric, positive definite

25

matrix T^{-1}, B is symmetric, positive definite matrix. Also, by
Propositions 1.6(a) and 1.1(c), $\alpha_+(B) \leq 2.$).

(4) Use DECOMP on the result of step (3) to obtain a minimal
decomposition for B.

(5) Use the algorithm to find $S_{22} = B^{-1}$.

(6) Use the 3-minus version of PRODUCT to get a decomposition of
$S_{21} = -S_{22}T_{21}T_{11}^{-1}$. Use the 5-minus version of PRODUCT to obtain a
decomposition of $T_{11}^{-1}T_{12}S_{22}T_{21}T_{11}^{-1}$, and append T_{11}^{-1} to get a
decomposition of S_{11}.

(7) Use (an obvious modification of) DECOMP on the results of steps
(5) and (6) to find a minimal (-)-decomposition of $S = T^{-1}$.

SECTION FOUR

In this section we present an analysis of the algorithm given in the
previous section. For simplicity we treat all arithmetic operations
identically and assume that the displacement ranks of the minors encountered
in the execution of the algorithm are maximal. Thus, the values given here
for computational complexity and storage requirements are upper bounds. It
seems likely, however, that these upper bounds in practice would be fairly
sharp. Since this algorithm has not been, and in its present form is likely
never to be, implemented, this is only speculation. In any event, the
analysis should demonstrate that this algorithm's demands in terms of storage
and computational complexity are too great for it to be practical in its
present form.

A.   COMPUTATION COUNT

Let $C(n)$ denote the number of operations needed to invert a symmetric,
positive definite, Toeplitz matrix T given in the form of a (+)-decomposition
of size 2. Then, from Section Three,

$$C(2^a) = 4\cdot\theta(2^a) \qquad\qquad \text{step (1)}$$

$$+ C(2^{a-1}) \qquad\qquad \text{step (2)}$$

$$+ 144\cdot\theta(2^{a-1}) \qquad\qquad \text{step (3)}$$

$$+ 126\cdot2^{a-1} \qquad\qquad \text{step (4) \quad (B has 58 terms)}$$

$$+ C(2^{a-1}) \qquad\qquad \text{step (5)}$$

$$+ (96 + 2016)\cdot\theta(2^{a-1}) \qquad\qquad \text{step (6)}$$

$$+ 4\cdot2^{a-1} + 1300\cdot2^{a-1} + \qquad\qquad \text{step (7)}$$

$$2\cdot\theta(2^{a-1}) + 650\cdot\theta(2^{a-1}) + \qquad\qquad \text{(by an analysis similar to}$$

$$6\cdot2^a \qquad\qquad \text{that given for DECOMP in §3)}$$

Thus $C(2^a)\approx 2C(2^{a-1}) + 2800\theta(2^{a-1}) + 1400\cdot2^{a-1}$. Now a FFT of a vector of

length $N=2^y$ requires $3N\cdot y/2$ operations. The convolutions of two vectors of

length 2^{a-1} require padding both vectors with zeroes to make them of length 2^a, two FFTs, 2^a multiplications and another FFT. Therefore, $\theta(2^{a-1}) = \frac{9}{2} \cdot 2^a \cdot a + 2^a$. Therefore, $C(2^a) \approx 2C(2^{a-1}) + (12,600a + 2,100)2^a$. (It is likely that these coefficients could be reduced if one required the matrix T to be real.)

Assuming that $C(1) = 1$, then

$$C(2^n) \approx \sum_{a=1}^{n} 2^{n-a} (12,600a + 2,100)2^a + 2^n$$

$$= 2^n(2,100 \cdot n + 12,600 \frac{(n+1)(n)}{2} + 1)$$

$$= 2^n(6,300 \cdot n^2 + 8,400 \cdot n + 1) .$$

That is, $C(N) \sim 6,300\ N(\log_2 N)^2 + 8,400\ N(\log_2 N) + N$.

Comparison with techniques which require $2N^2$ operations to invert symmetric Toeplitz matrices show that the algorithm given here is as fast as these when $3,150(\log_2 N)^2 + 4,200(\log_2 N) = N$, or for $N \geq 2^{21} \approx 2 \times 10^6$.

B. STORAGE REQUIREMENTS

It is obvious that the maximum amount of storage is required by the algorithm following the last execution of step (6). The principal amount of storage is required to hold the decompositions of S_{11} and S_{21}. This amount is $684 \cdot n$ numbers, where T is an n x n matrix.

To conclude, the algorithm presented in Section Three for inversion of symmetric, positive definite, N x N Toeplitz matrices:

(1) Has computational complexity asymptotic to $6,300\ N(\log_2 N)^2$.
(2) Has storage requirements of at least $684 \cdot N$.

REFERENCES

1. Bitmead, Robert R., and Anderson, Brian D.O., "Asymptotically Fast
    Solution of Toeplitz and Related Systems of Linear Equations,"
    Linear Algebra and its Applications, 34, pp 103-116, 1980.

2. Kailath, Thomas, Kung, Sun-Yuan, and Morf, Martin, "Displacement Ranks of
    Matrices and Linear Equations," Journal of Mathematical Analysis
    and Applications, 68, pp 395-407, 1979.